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The magnetoelastic problem of cracks in bonded dissimilar materials

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Abstract

In this paper we consider the magnetoelastic problem of straight cracks lying along interface of two dissimilar soft ferromagnetic materials subjected to remote uniform magnetic induction. Based on the Hilbert problem formulation and the technique of analytical continuation, closed form solution for magnetic fields and both the magnetoelastic stresses and the Maxwell stresses are obtained. It is found that the singularity of magnetoelastic stresses has similar trig-log character as those obtained for pure elastic problems which were free from the discontinuous jumps of the magnetic properties and fields across the interface. For illustrating the use of present approach, the detailed results for a single crack case are provided and verified by comparison with the existing ones under special cases. The numerical examples of magnetoelastic stress distribution for different material properties are presented graphically. Expressions of the stress intensity factors in the vicinity of crack tip and crack opening condition are also derived. It is shown that the crack open assumption is valid except a limiting range of distance measured from the crack tip. © 2002 Published by Elsevier Science Ltd.

Keywords: Trig-log character; Critical incident angle

1. Introduction

Due to the rapidly increasing use of composite materials in advanced engineering structure, the damage tolerance and reliability for the composite structures have been matters of concern. There arose the problem of finding the stress distribution in bonded dissimilar materials with cracks on the interface. The elastic problems of straight cracks between dissimilar media under in-plane load and bending have been studied by England (1965), Rice and Sih (1965), Sih and Rice (1964). They found the stresses near the tips of straight cracks between dissimilar materials possess trig-log singularity. The fracture mechanics on the tips of interfacial cracks was discussed by Rice (1988). All the above investigators have focused on the interfacial crack problems with mechanical type of source. Nevertheless, it is still a challenging and interesting study to determine the magnetic and magnetoelastic fields for two dissimilar materials containing interfacial cracks subjected to magnetic loading. The general theory of magnetoelastic interactions was developed by Tiersten

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(1964, 1965) and Brown (1966). Several investigators, such as Pao and Yeh (1973), Eringen and Maugin (1989), have deduced physical models and applications of the magnetoelastic interaction.

For the magnetelastic problems of crack, Shindo (1977, 1980) derived stress intensity factor near the crack tips and Asanyan (1988) studied the interfacial crack problem based on the linear theory by Pao and Yeh (1973) and integral transformation. Sabir and Maugin (1996), Fomethe and Maugin (1998) provided the expression of driving force on the crack tips for soft and hard ferromagnets. The merits of complex variable method to deal the crack problems have been indicated by Muskhelishvili (1953). This method is efficient in studying crack problems not only for elastic fields but also for magnetoelastic fields. The author used the complex variable technique to find the magnetic fields and magnetoelastic stresses distribution of a soft ferromagnetic material containing a straight crack (Lin and Yeh, 2002).

In the present study, we aim to find the general solution of the magnetoelastic problem with straight cracks in bonded dissimilar materials. Based upon the technique of complex variable, such as analytic continuation, the magnetic fields and the magnetoelastic stress functions in each material are obtained in a closed form. An explicit form of solution is given for a single line crack lying in the interface of bi-material plate under remote uniform magnetic induction. The stress intensity factors are also provided to present the singular behavior in the vicinity of crack tip. All the solutions derived here become invalid under the condition of crack close. The explicit form of expression for the crack open condition is given to find the critical incident angle of magnetic induction. Variations of magnetoelastic stresses on several parameters are displayed graphically to illustrate the use of this paper.

2. Magnetic fields around the interfacial cracks

Two homogeneous, ferromagnetic materials occupy the upper half plane S^+ and lower half plane S^-. As shown in Fig. 1, the magnetic and elastic properties of the material in S^+ and S^- are marked by subscripts 1 and 2, respectively. In which, μ_{rj} ($j = 1, 2$) is relative magnetic permeability and λ_j, G_j denote Lamé's constants in the corresponding area. As mentioned by Moon (1984), the relative magnetic permeability μ_{rj} of linear soft ferromagnetic materials have order of magnitude 10^2–$10^5 \gg 1$. If there are straight cracks lying on the interface of two materials, the imperfect bonded interface can be represented as the sum of L and L^* as indicated in Fig. 1. Here $L = L_1 + L_2 + \cdots + L_n$ is the union of n straight cracks on $L_k = (p_k, q_k)$ and L^* is union of the rest bonded area. Let the interface be situated on the real axis of the complex plane z

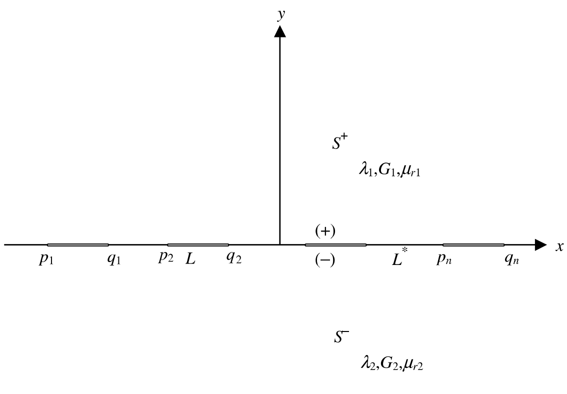

Fig. 1. The line cracks on the partially bonded interface between two dissimilar media.

($= x + \mathrm{i}y$) and t be the points located on it. The magnetic induction $(B_y^+)_1$ and $(B_y^-)_2$ are specified on the upper and lower surfaces of L, i.e.

$$(B_y^+)_1 = b^+(t) \quad \text{on } L \tag{1}$$

$$(B_y^-)_2 = b^-(t) \quad \text{on } L \tag{2}$$

Furthermore, the boundary conditions of magnetic field (Moon, 1984)

$$\oint_C (H_x\,\mathrm{d}x + H_y\,\mathrm{d}y) = 0, \qquad \int_S (B_x n_x + B_y n_y)\,\mathrm{d}s = 0 \tag{3}$$

lead the continuity of magnetic on bounded interface L^* as

$$(H_x)_1 = (H_x)_2 \quad \text{on } L^* \tag{4}$$

$$(B_y)_1 = (B_y)_2 \quad \text{on } L^* \tag{5}$$

with

$$(B_k)_j = \mu_0 \mu_{rj} (H_k)_j \quad k = x, y \text{ and } j = 1, 2 \tag{6}$$

where the symbols B_j, H_j and μ_0 ($= 4\pi \times 10^{-7}$ N/A^2) are magnetic induction, magnetic intensity and a universal constant, respectively. Those quantities with superscripts + and − are approached from S^+ and S^-. According to the detailed derivations given by Lin and Yeh (2002), the magnetic boundary conditions in Eqs. (1), (2), (4) and (5) can be expressed in terms of the complex functions $\Phi_j^M(z)$ and $\Omega_j^M(z)$ ($j = 1, 2$) as

$$\Phi_1^{M+}(t) - \Omega_1^{M-}(t) = -2\mathrm{i}\frac{b^+(t)}{\mu_0 \mu_{r_1}} \quad \text{on } L \tag{7}$$

$$\Phi_2^{M-}(t) - \Omega_2^{M+}(t) = -2\mathrm{i}\frac{b^-(t)}{\mu_0 \mu_{r_2}} \quad \text{on } L \tag{8}$$

and

$$\Phi_1^M(t) + \Omega_1^M(t) = \Phi_2^M(t) + \Omega_2^M(t) \quad \text{on } L^* \tag{9}$$

$$\mu_0 \mu_{r_1}[\Phi_1^M(t) - \Omega_1^M(t)] = \mu_0 \mu_{r_2}[\Phi_2^M(t) - \Omega_2^M(t)] \quad \text{on } L^* \tag{10}$$

where

$$\Phi_j^M(z) = h_j'(z), \qquad \Omega_j^M(z) = \overline{\Phi_j^M}(z) \quad j = 1, 2 \tag{11}$$

with

$$h_j'(z) = (H_x - \mathrm{i}H_y)_j \tag{12}$$

Those quantities with superscript M are related to magnetic fields. The notation $\overline{\Phi_j^M}(z)$ denotes complex conjugate of the coefficients (not argument) in $\Phi_j^M(z)$. Since Eqs. (9) and (10) may be regarded as the conditions of analytic continuation of $\Phi_j^M(t)$ and $\Omega_j^M(t)$, the functions $\Phi_1^M(t)$ and $\Omega_1^M(t)$ can be solved explicitly in terms of $\Phi_2^M(t)$ and $\Omega_2^M(t)$ as

$$\Phi_1^M(t) = \frac{\mu_{r_1} + \mu_{r_2}}{2\mu_{r_1}}\Phi_2^M(t) + \frac{\mu_{r_1} - \mu_{r_2}}{2\mu_{r_1}}\Omega_2^M(t) \tag{13}$$

$$\Omega_1^M(t) = \frac{\mu_{r_1} - \mu_{r_2}}{2\mu_{r_1}}\Phi_2^M(t) + \frac{\mu_{r_1} + \mu_{r_2}}{2\mu_{r_1}}\Omega_2^M(t) \tag{14}$$

which are valid in everywhere of z-plane. On adding and subtracting of Eqs. (7) and (8) and applying Eqs. (13) and (14) we have

$$[\Phi_2^M(t) - \Omega_2^M(t)]^+ + [\Phi_2^M(t) - \Omega_2^M(t)]^- = f^M(t) \tag{15}$$

$$[\Phi_2^M(t) + \Omega_2^M(t)]^+ - [\Phi_2^M(t) + \Omega_2^M(t)]^- = g^M(t) \tag{16}$$

The symbols $f^M(t)$ and $g^M(t)$ are in form as

$$f^M(t) = \frac{-4\mathrm{i}\mu_{r_1}}{\mu_0(\mu_{r_1} + \mu_{r_2})}\left[\frac{b^+(t)}{\mu_{r_1}} + \frac{b^-(t)}{\mu_{r_2}}\right] \tag{17}$$

$$g^M(t) = \frac{-4\mathrm{i}}{\mu_0(\mu_{r_1} + \mu_{r_2})}[b^+(t) - b^-(t)] \tag{18}$$

which must satisfy the Hölder condition on L. Since Eq. (15) is a non-homogeneous Hilbert problem for the function $\Phi_2^M(z) - \Omega_2^M(z)$ and Eq. (16) is a Plemelj equation for the function $\Phi_2^M(z) + \Omega_2^M(z)$, their solutions can be obtained as

$$\Phi_2^M(z) - \Omega_2^M(z) = \frac{X^M(z)}{2\pi\mathrm{i}}\int_L \frac{f^M(t)}{X^{M+}(t)(t-z)}\mathrm{d}t + X^M(z)Q_n(z) \tag{19}$$

$$\Phi_2^M(z) + \Omega_2^M(z) = \frac{1}{2\pi\mathrm{i}}\int_L \frac{g^M(t)}{(t-z)}\mathrm{d}t + d_0 \tag{20}$$

where the Plemelj function satisfying $X^{M+}(t) = -X^{M-}(t)$ on L will be

$$X^M(z) = \prod_{j=1}^{n}(z-p_j)^{-1/2}(z-q_j)^{-1/2} \tag{21}$$

with the necessary branch cuts and the branch selected such that

$$\lim_{z\to\infty}[z^n X^M(z)] = 1 \tag{22}$$

Eq. (21) implies that the near-tip magnetic induction always possesses the inverse square root singularity in terms of the distance away from the crack tip. This feature would not be affected by the discontinuity of magnetic permeability jumping across the material interface. The symbol d_0 is a constant to be holomorphic in the whole plane and the function $Q_n(z)$ is a polynomial of degree not greater than n, i.e.

$$Q_n(z) = \sum_{j=0}^{n} c_j z^j \tag{23}$$

By the use of Eqs. (13), (14), (19) and (20), the general solutions of $\Phi_j^M(z)$ and $\Omega_j^M(z)$ ($j = 1, 2$) can be expressed in a compact form as

$$\Phi_2^M(z) = \frac{1}{4\pi\mathrm{i}}\int_L \frac{g^M(t)}{(t-z)}\mathrm{d}t + \frac{X^M(z)}{4\pi\mathrm{i}}\int_L \frac{f^M(t)}{X^{M+}(t)(t-z)}\mathrm{d}t + \frac{1}{2}d_0 + \frac{1}{2}X^M(z)Q_n(z) \tag{24}$$

$$\Omega_2^M(z) = \frac{1}{4\pi\mathrm{i}}\int_L \frac{g^M(t)}{(t-z)}\mathrm{d}t - \frac{X^M(z)}{4\pi\mathrm{i}}\int_L \frac{f^M(t)}{X^{M+}(t)(t-z)}\mathrm{d}t + \frac{1}{2}d_0 - \frac{1}{2}X^M(z)Q_n(z) \tag{25}$$

and

$$\Phi_1^M(z) = \frac{1}{4\pi i}\int_L \frac{g^M(t)}{(t-z)}\,dt + \frac{\mu_{r_2}}{4\mu_{r_1}}\frac{X^M(z)}{\pi i}\int_L \frac{f^M(t)}{X^{M+}(t)(t-z)}\,dt + \frac{1}{2}d_0 + \frac{\mu_{r_2}}{2\mu_{r_1}}X^M(z)Q_n(z) \tag{26}$$

$$\Omega_1^M(z) = \frac{1}{4\pi i}\int_L \frac{g^M(t)}{(t-z)}\,dt - \frac{\mu_{r_2}}{4\mu_{r_1}}\frac{X^M(z)}{\pi i}\int_L \frac{f^M(t)}{X^{M+}(t)(t-z)}\,dt + \frac{1}{2}d_0 - \frac{\mu_{r_2}}{2\mu_{r_1}}X^M(z)Q_n(z) \tag{27}$$

Since the values of relative magnetic permeability μ_{r_1} and μ_{r_2} for both soft ferromagnetic materials in S^+ and S^- are much higher than that of the air enclosed by cracks, the upper and lower boundary of cracks may be viewed as insulated surfaces as noted by Lin and Yeh (2002). Thus

$$b^+(t) = b^-(t) = 0, \quad f^M(t) = g^M(t) = 0 \tag{28}$$

Upon the using of Eqs. (11) and (12) and the magnetic induction $\mathbf{B}_0 = B_{0x} + iB_{0y}$ applied at the infinity of S^-, the functions $\Phi_2^M(z)$ and $\Omega_2^M(z)$ for large value of $|z|$ take the form as

$$\Phi_2^M(z) = \Gamma^M + O\left(\frac{1}{z}\right), \qquad \Omega_2^M(z) = \overline{\Gamma^M} + O\left(\frac{1}{z}\right) \quad \text{for } z \gg 1 \tag{29}$$

where

$$\Gamma^M = \frac{1}{\mu_0\mu_{r_2}}(B_{0x} - iB_{0y}) \tag{30}$$

By substituting Eqs. (22) and (28) into (24) and (25) then comparing with Eq. (29) yield

$$c_n = \Gamma^M - \overline{\Gamma^M} = \frac{-2iB_{0y}}{\mu_0\mu_{r_2}}, \qquad d_0 = \Gamma^M + \overline{\Gamma^M} = \frac{2B_{0x}}{\mu_0\mu_{r_2}} \tag{31}$$

The remaining n unknowns $c_0, c_1, \ldots, c_{n-1}$ in the polynomial $P_n(z)$ can be found by applying the first part of Eq. (3) on the contours surrounding each crack L_j. Through the use of Eqs. (11) and (12), such a requirement can be formulated as

$$\int_{L_j}[\Phi_2^{M+}(t) + \Omega_2^{M-}(t)]\,dt - \int_{L_j}[\Phi_1^{M-}(t) + \Omega_1^{M+}(t)]\,dt = 0 \tag{32}$$

Alternatively, we can use Eqs. (13) and (14) to rearrange Eq. (32) in the form

$$\int_{L_j}\left\{[\Phi_2^M(t) - \Omega_2^M(t)]^+ - [\Phi_2^M(t) - \Omega_2^M(t)]^-\right\}dt = 0 \tag{33}$$

This is a system of n linear equations which can be used to determine the n unknowns $c_0, c_1, \ldots, c_{n-1}$. On the basis of unique theorem (Muskhelishvili, 1953), the coefficients of z in Eq. (23) are obtained from these conditions uniquely. Once the magnetic boundary conditions are specified on the crack surfaces, the general solution to the present problem is reduced to the evaluation of singular integrals with Cauchy-type kernels.

For illustrating the use of above derivation, we consider a crack lying within the range $(-a, a)$ on the interface as depicted in Fig. 2. By taking $n = 1$ and $(p_1, q_1) = (-a, a)$, the Plemelj function $X^M(z)$ can be obtained as

$$X^M(z) = \frac{1}{\sqrt{z^2 - a^2}} \tag{34}$$

In Eq. (31), the coefficient d_0 remains unchanged but c_n reduces to c_1 i.e.

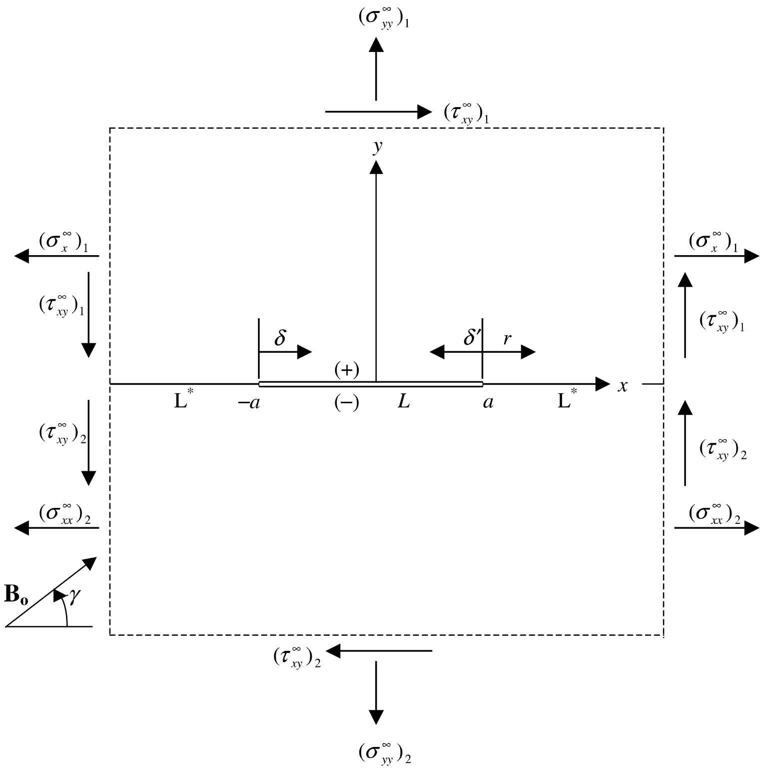

Fig. 2. The far field stresses and magnetic induction on bonded soft ferromagnetic solids with single line crack.

$$c_1\Gamma^M - \overline{\Gamma^M} = \frac{-2\mathrm{i}B_{0y}}{\mu_0\mu_{r_2}} \tag{35}$$

Substituting Eqs. (24), (25), (28) into (33) renders

$$c_0 = 0 \tag{36}$$

After determining all the coefficients in Eqs. (24) and (25), the complex functions $\Phi_1(z)$ and $\Phi_2(z)$ take the explicit form

$$h_1'(z) = \Phi_1^M(z) = \frac{1}{\mu_0\mu_{r_1}}\left(\frac{\mu_{r_1}}{\mu_{r_2}}B_{0x} - \frac{\mathrm{i}B_{0y}z}{\sqrt{z^2-a^2}}\right) \tag{37}$$

$$h_2'(z) = \Phi_2^M(z) = \frac{1}{\mu_0\mu_{r_2}}\left(B_{0x} - \frac{\mathrm{i}B_{0y}z}{\sqrt{z^2-a^2}}\right) \tag{38}$$

Hence, the magnetic fields are obtained from Eqs. (6), (11) and (12) as

$$(H_x + \mathrm{i}H_y)_1 = \frac{1}{\mu_0\mu_{r_1}}(B_x + \mathrm{i}B_y)_1 = \frac{1}{\mu_0\mu_{r_1}}\left(\frac{\mu_{r_1}}{\mu_{r_2}}B_{0x} + \frac{\mathrm{i}B_{0y}\bar{z}}{\sqrt{\bar{z}^2-a^2}}\right) \tag{39}$$

in S^+ and

$$(H_x + iH_y)_2 = \frac{1}{\mu_0\mu_{r_2}}(B_x + iB_y)_2 = \frac{1}{\mu_0\mu_{r_2}}\left(B_{0x} + \frac{iB_{0y}\bar{z}}{\sqrt{\bar{z}^2 - a^2}}\right) \tag{40}$$

in S^-. It is seen that the magnetic field possess the square root singularity in terms of the distance r measured from the tips of crack. Basing on this singular behavior, it is convenient to define the magnetic flux intensity factor to quantify the intensification of magnetic energy in the vicinity of the crack tip as

$$k^M = \lim_{r\to 0}\sqrt{2r}B \tag{41}$$

where the net magnetic flux B is given by

$$B = \sqrt{(B_x^2 + B_y^2)_j} \quad (j = 1, 2) \tag{42}$$

Notice that the streamlines of magnetic flux are repelled by cracks with boundary as insulted surfaces for magnetic fields but will be attracted by ferromagnetic media when applied from air. Therefore, the features of magnetic fields which are uniform in a thin body as obtained by van de Ven (1984) are quite different from that in Eqs. (39) and (40). Nevertheless, the magnetic fields in the problem of elliptic inclusion will be provided in the future study that covers both cases and can be used to confirm the accuracy of their results.

Substituting Eqs. (39), (40) and (42) into (41) and letting $z = a + r$, we have

$$k^M = B_{0y}\sqrt{a} \tag{43}$$

For the special case of homogeneous material ($\mu_{r_1} = \mu_{r_2}$), the solutions (39) and (40) are degenerated to the same form and are in accordance to the corresponding homogeneous material problem given by Lin and Yeh (2002). It is reasonable that the magnetic flux intensity factor disappears (i.e. $k^M = 0$) for the problem of perfect bonding by letting $a = 0$.

3. Formulations of magnetoelastic fields

For the interface between two materials, the tractions t_{yy}^T and t_{xy}^T will be specified on L, while the stresses and displacement are required to be continuous on L^*, i.e.

$$(t_{yy}^{T+})_1 - i(t_{xy}^{T+})_1 = P^+(t) \quad \text{on } L \tag{44}$$

$$(t_{yy}^{T-})_2 - i(t_{xy}^{T-})_2 = P^-(t) \quad \text{on } L \tag{45}$$

and

$$(t_{yy}^{T})_1 - i(t_{xy}^{T})_1 = (t_{yy}^{T})_2 - i(t_{xy}^{T})_2 \quad \text{on } L^* \tag{46}$$

$$(u_x + iu_y)_1 = (u_x + iu_y)_2 \quad \text{on } L^* \tag{47}$$

Here the superscript T is used to denote the total stresses. Extending the stress combinations given by Lin and Yeh (2002) to the regions S^+ and S^-, it follows that

$$(t_{xx} + t_{yy})_j^T = (t_{xx} + t_{yy})_j + (t_{xx} + t_{yy})_j^M \tag{48}$$

and

$$(t_{yy} - it_{xy})_j^T = (t_{yy} - it_{xy})_j + (t_{yy} - it_{xy})_j^M \tag{49}$$

where

$$
\begin{aligned}
&(t_{xx}+t_{yy})_j = 2[\Phi_j(z)+\overline{\Phi_j(z)}] + \mu_0\chi_j h'_j(z)\overline{h'_j(z)} \\
&(t_{xx}+t_{yy})_j^M = \mu_0\chi_j h'_j(z)\overline{h'_j(z)} \\
&(t_{yy}-\mathrm{i}t_{xy})_j = \Phi_j(z)+\Omega_j(\bar{z})+(z-\bar{z})\Phi'_j(z)+\frac{\mu_0}{2}[\chi_j h'_j(z)-(\chi_j-1)\overline{h'_j(z)}]\overline{h'_j(z)} \\
&(t_{yy}-\mathrm{i}t_{xy})_j^M = \frac{\mu_0}{2}[\chi_j h'_j(z)-\mu_{rj}\overline{h'_j(z)}]\overline{h'_j(z)}
\end{aligned}
\tag{50}
$$

where $j = 1, 2$ for $z \in S^+, S^-$ and the functions $\Phi_j(z)$ and $\Omega_j(z)$ are defined as

$$\Phi_j(z) = \phi'_j(z) \tag{51}$$

$$\Omega_j(z) = \Phi_j(z) + z\overline{\Phi'_j(z)} + \overline{\psi'_j(z)} - \frac{\mu_0}{2}\overline{h'_j}(z)\overline{h'_j}(z) \tag{52}$$

In Eqs. (48) and (49), the total stresses are the sum of Maxwell stress with superscript M and magnetoelastic stresses. It is noted that the body force terms are dropped in Eq. (50). The corresponding displacement in the absence of body force terms can be expressed as

$$2G(u_x+\mathrm{i}u_y)_j = \kappa_j\phi_j(z) - z\overline{\phi'_j(z)} - \overline{\psi_j(z)} - \frac{G_j}{(\lambda_j+2G_j)}\mu_0\chi_j\int h'_j(z)\overline{h'_j(z)}\,\mathrm{d}z \tag{53}$$

where

$$\kappa_j = \frac{\lambda_j+3G_j}{\lambda_j+G_j} \tag{54}$$

The notation χ_j $(= \mu_{rj}-1, j = 1, 2)$ are magnetic susceptibility in both half-plane and the symbols u_x and u_y are displacements along x and y directions. Using Eqs. (7), (8), (11), (48)–(54) and taking $t = \bar{t}$, Eqs. (44)–(47) can be written in form as

$$\Phi_1^+(t) + \Omega_1^-(t) = P^+(t) + 2\mathrm{i}\frac{\chi_1}{\mu_{r_1}}b^+(t)\overline{h'_1}^-(t) \quad \text{on } L \tag{55}$$

$$\Phi_2^-(t) + \Omega_2^+(t) = P^-(t) + 2\mathrm{i}\frac{\chi_2}{\mu_{r_2}}b^-(t)\overline{h'_2}^+(t) \quad \text{on } L \tag{56}$$

and

$$\Phi_1(t) + \Omega_1(t) = \Phi_2(t) + \Omega_2(t) + A_{12}(t) \quad \text{on } L^* \tag{57}$$

$$\frac{1}{G_1}[\kappa_1\Phi_1(t) - \Omega_1(t)] = \frac{1}{G_2}[\kappa_2\Phi_2(t) - \Omega_2(t)] + B_{12}(t) \quad \text{on } L^* \tag{58}$$

where

$$A_{12}(t) = \mu_0\chi_2[h'_2(t) - \overline{h'_2}(t)]\overline{h'_2}(t) - \mu_0\chi_1[h'_1(t) - \overline{h'_1}(t)]\overline{h'_1}(t) \tag{59}$$

$$B_{12}(t) = \frac{\mu_0}{2G_1}\left[\frac{2G_1\chi_1}{\lambda_1+2G_1}h'_1(t) + \overline{h'_1}(t)\right]\overline{h'_1}(t) - \frac{\mu_0}{2G_2}\left[\frac{2G_2\chi_2}{\lambda_2+2G_2}h'_2(t) + \overline{h'_2}(t)\right]\overline{h'_2}(t) \tag{60}$$

In Eq. (58), only the derivatives of displacement with t are required to be continuous across L^*, i.e.

$$\left(\frac{\partial u_x}{\partial t}+\mathrm{i}\frac{\partial u_y}{\partial t}\right)_1=\left(\frac{\partial u_x}{\partial t}+\mathrm{i}\frac{\partial u_y}{\partial t}\right)_2 \tag{61}$$

The functions $\Phi_1(t)$ and $\Omega_1(t)$ can be solved explicitly in terms of $\Phi_2(t)$ and $\Omega_2(t)$ as

$$\Phi_1(t)=\frac{G_2+\kappa_2G_1}{G_2(1+\kappa_1)}\Phi_2(t)+\frac{G_2-G_1}{G_2(1+\kappa_1)}\Omega_2(t)+\frac{1}{(1+\kappa_1)}[A_{12}(t)+G_1B_{12}(t)] \tag{62}$$

$$\Omega_1(t)=\frac{\kappa_1G_2-\kappa_2G_1}{G_2(1+\kappa_1)}\Phi_2(t)+\frac{\kappa_1G_2+G_1}{G_2(1+\kappa_1)}\Omega_2(t)+\frac{1}{(1+\kappa_1)}[\kappa_1A_{12}(t)-G_1B_{12}(t)] \tag{63}$$

these equations are valid everywhere in the z-plane and can be substituted into Eqs. (55) and (56) to obtain

$$[\Phi_2(t)+\Omega_2(t)]^+ +\alpha[\Phi_2(t)+\Omega_2(t)]^- =f(t) \tag{64}$$

$$[\Phi_2(t)-\alpha\Omega_2(t)]^+ -[\Phi_2(t)-\alpha\Omega_2(t)]^- =g(t) \tag{65}$$

where

$$\begin{aligned}f(t)={}&\frac{G_2(1+\kappa_1)}{G_2+\kappa_2G_1}\left[P^+(t)+2\mathrm{i}\frac{\chi_1}{\mu_{r_1}}b^+(t)\overline{h_1'}^-(t)\right]+\frac{G_1(1+\kappa_2)}{G_2+\kappa_2G_1}\left[P^-(t)+\frac{\chi_2}{\mu_{r_2}}b^-(t)\overline{h_2'}^+(t)\right]\\&-\frac{G_2}{G_2+\kappa_2G_1}\{A_{12}^+(t)\kappa_1A_{12}^-(t)+G_1[B_{12}^+(t)-B_{12}^-(t)]\}\end{aligned} \tag{66}$$

$$\begin{aligned}g(t)={}&\frac{G_2(1+\kappa_1)}{G_2+\kappa_2G_1}\left\{P^+(t)-P^-(t)+2\mathrm{i}\left[\frac{\chi_1}{\mu_{r_1}}b^+(t)\overline{h_1'}^-(t)-\frac{\chi_2}{\mu_{r_2}}b^-(t)\overline{h_2'}^+(t)\right]\right\}\\&-\frac{G_2}{G_2+\kappa_2G_1}\{A_{12}^+(t)+\kappa_1A_{12}^-(t)+G_1[B_{12}^+(t)-B_{12}^-(t)]\}\end{aligned} \tag{67}$$

must satisfy the Hölder condition on L. The parameter α is

$$\alpha=\frac{G_1+\kappa_1G_2}{G_2+\kappa_2G_1} \tag{68}$$

4. Solutions of magnetoelastic fields

Knowing that Eq. (64) is a non-homogeneous Hilbert equation for $\Phi_2(t)+\Omega_2(t)$ and Eq. (65) is a Plemelj equation for $\Phi_2(t)-\alpha\Omega_2(t)$, they have solutions as

$$\Phi_2(t)+\Omega_2(t)=\frac{X(z)}{2\pi\mathrm{i}}\int_L\frac{f(t)}{X^+(t)(t-z)}\mathrm{d}t+X(z)R_n(z) \tag{69}$$

$$\Phi_2(t)-\alpha\Omega_2(t)=\frac{1}{2\pi\mathrm{i}}\int_L\frac{g(t)}{t-z}\mathrm{d}t+e_0 \tag{70}$$

where the Plemelj function satisfying $X^+(t)=-\alpha X^-(t)$ can be expressed as

$$X(z)=\prod_{j=1}^{n}(z-p_j)^{-1/2+\mathrm{i}\beta}(z-q_j)^{-1/2-\mathrm{i}\beta} \tag{71}$$

which provides the necessary branch cut then is selected such that

$$\lim_{z\to\infty}[z^n X(z)] = 1 \tag{72}$$

In Eq. (71), the exponent β is defined as

$$\beta = \frac{1}{2\pi}\log\alpha \tag{73}$$

The function $R_n(z)$ is a polynomial of z with degree not greater than n, i.e.

$$R_n(z) = \sum_{j=0}^{n} s_j z^j \tag{74}$$

and e_0 that is holomorphic everywhere in the z-plane is a constant. By applying Eqs. (62), (63), (68)–(70), the general solutions of the four unknown functions $\Phi_j(t)$, $\Omega_j(t)$, $(j = 1, 2)$ can be rearranged into a compact form

$$\Phi_1(z) = \frac{(G_2 + G_1\kappa_2)[G_2(1+\kappa_1)F_1(z) + G_1(1+\kappa_2)F_2(z)]}{G_2(1+\kappa_1)[G_1(1+\kappa_2) + G_2(1+\kappa_1)]} + \frac{A_{12}(z) + G_1 B_{12}(z)}{(1+\kappa_1)} \tag{75}$$

$$\Omega_1(z) = \frac{G_2(1+\kappa_1)(G_1 + G_2\kappa_1)F_1(z) - G_1(1+\kappa_2)(G_2 + G_1\kappa_2)F_2(z)}{G_2(1+\kappa_1)[G_1(1+\kappa_2) + G_2(1+\kappa_1)]} + \frac{\kappa_1 A_{12}(z) - G_1 B_{12}(z)}{(1+\kappa_1)} \tag{76}$$

and

$$\Phi_2(z) = \frac{(G_1 + G_2\kappa_1)F_1(z) + (G_2 + G_1\kappa_2)F_2(z)}{G_1(1+\kappa_2) + G_2(1+\kappa_1)} \tag{77}$$

$$\Omega_2(z) = \frac{(G_2 + G_1\kappa_2)[F_1(z) - F_2(z)]}{G_1(1+\kappa_2) + G_2(1+\kappa_1)} \tag{78}$$

where

$$F_1(z) = \frac{X(z)}{2\pi i}\int_L \frac{f(t)}{X^+(t)(t-z)}\,dt + X(z)R_n(z) \tag{79}$$

$$F_2(z) = \frac{1}{2\pi i}\int_L \frac{g(t)}{t-z}\,dt + e_0 \tag{80}$$

and $A_{12}(z)$ and $B_{12}(z)$ are obtained from Eqs. (59) and (60) by letting $t = z$ in those equations.

Following the procedure provided by Lin and Yeh (2002) for homogeneous material with a straight crack, the functions $\Phi_2(z)$ and $\Omega_2(z)$ at infinity take the form as

$$\Phi_2(z) = \Gamma + \frac{1}{2\mu_0}\left(\frac{1}{4} - \frac{\chi_2}{\mu_{r_2}^2}\right)(B_{0x}^2 + B_{0y}^2) + O\left(\frac{1}{z}\right) \quad \text{for } |z| \gg 1 \tag{81}$$

$$\Omega_2(z) = \overline{\Gamma} + \overline{\Gamma'} - \frac{1}{2\mu_0}\left[\left(\frac{1}{4} - \frac{\chi_2}{\mu_{r_2}^2}\right)(B_{0x}^2 + 4iB_{0x}B_{0y} - 3B_{0y}^2)\right] + O\left(\frac{1}{z}\right) \quad \text{for } |z| \gg 1 \tag{82}$$

where

$$\Gamma = \frac{1}{4}(\sigma_{xx}^\infty + \sigma_{yy}^\infty)_2 + i\frac{2G_2\omega_2^\infty}{1+\kappa_2}, \qquad \Gamma' = -\frac{1}{2}(\sigma_{xx}^\infty - \sigma_{yy}^\infty - 2i\tau_{xy}^\infty)_2 \tag{83}$$

here the symbols $(\sigma_{xx}^{\infty})_2$, $(\sigma_{yy}^{\infty})_2$ and $(\tau_{xy}^{\infty})_2$ are the normal stresses along x and y directions and shear stress at infinity and ω_2^{∞} denotes the rotation at infinity in S^- as referred to Fig. 2. From the viewpoint of force equivalent, the stress components σ_{yy}^{∞} and τ_{xy}^{∞} are continuous across the interface, i.e. $(\sigma_{yy}^{\infty})_1 = (\sigma_{yy}^{\infty})_2$, $(\tau_{xy}^{\infty})_1 = (\tau_{xy}^{\infty})_2$, but the component σ_{xx}^{∞} is not. Since the component σ_{xx}^{∞} may jump across the interface (i.e. x-axis), we now express Γ and Γ' in terms of the stresses along x and y directions rather than the principal stresses σ_1^{∞} and σ_2^{∞} given by Lin and Yeh (2002). Nevertheless, both expressions will be coincident for the homogeneous material case.

The coefficient s_n in Eq. (74) and constant e_0 in Eq. (70) can be found by applying the Eqs. (81) and (82) at infinity. In addition, the remaining n unknowns $s_0, s_1, \dots, s_{n-1}$ in the polynomial $R_n(z)$ can be determined from the conditions that the displacements must be single valued, i.e., the displacement must revert to its original values while the point z describes a contour around a given segment, say L_j of jth crack. In order to express such a requirement in a mathematical form, we take the derivative of displacement on the upper surfaces of cracks and use Eqs. (51)–(53) to obtain

$$[u_x'(t) + \mathrm{i}u_y'(t)]_1^+ = \frac{1}{2G_1}\left\{\left[\kappa_1\Phi_1^+(t) - \Omega_1^-(t) - \frac{\mu_0}{2}\overline{h_1'}^-(t)\overline{h_1'}^-(t) - \frac{G_1}{\lambda_1 + 2G_1}\mu_0\chi_1 h_1'^+(t)\overline{h_1'}^+(t)\right]\right\} \quad \text{for } t \in L \tag{84}$$

Similarly, the derivative of displacement $[u_x'(t) + \mathrm{i}u_y'(t)]_2^-$ in lower surface also can be obtained by replacing 1, + and − with 2, − and + in Eq. (84). Thus, the requirement that the displacement must be single valued is equivalent to

$$\begin{aligned}&\frac{1}{2G_1}\int_{L_j}\left[\kappa_1\Phi_1^+(t) - \Omega_1^-(t) - \frac{\mu_0}{2}\overline{h_1'}^-(t)\overline{h_1'}^-(t) - \frac{G_1}{\lambda_1 + 2G_1}\mu_0\chi_1 h_1'^+(t)\overline{h_1'}^+(t)\right]\mathrm{d}t \\ &\quad - \frac{1}{2G_2}\int_{L_j}\left[\kappa_2\Phi_2^-(t) - \Omega_2^+(t) - \frac{\mu_0}{2}\overline{h_2'}^+(t)\overline{h_2'}^+(t) - \frac{G_2}{\lambda_2 + 2G_2}\mu_0\chi_2 h_2'^-(t)\overline{h_2'}^-(t)\right]\mathrm{d}t = 0\end{aligned} \tag{85}$$

for $j = 1, 2, \dots, n$. The substitution of Eqs. (62) and (63) into (85) and the use of Eqs. (7), (8), (11) and (28) yield

$$\begin{aligned}&\int_{L_j}\Big\{[\kappa_1(G_2 + \kappa_2 G_1)[\Phi_2^+(t) - \Phi_2^-(t)] + (G_1 + \kappa_1 G_2)[\Omega_2^+(t) - \Omega_2^-(t)] \\ &\quad + \frac{\mu_0 G_1}{2}[\overline{h_2'}^+(t)\overline{h_2'}^+(t) - \overline{h_2'}^-(t)\overline{h_2'}^-(t)] + \frac{\mu_0 G_2\kappa_1}{2}[\overline{h_1'}^+(t)\overline{h_1'}^+(t) - \overline{h_1'}^-(t)\overline{h_1'}^-(t)]\Big\}\mathrm{d}t = 0\end{aligned} \tag{86}$$

This is a system of n linear equation which can be used to solve the n unknowns $s_0, s_1, \dots, s_{n-1}$.

Basing on the unique theorem (Muskhelishvili, 1953), these conditions determine the coefficients of z in Eq. (74) uniquely.

For the illustrating case of a single crack lying within the range $(-a, a)$ on the interface as shown in Fig. 2, we take $n = 1$ and $(p_1, q_1) = (-a, a)$ on Eq. (71) to obtain the Plemelj function $X(z)$ as

$$X(z) = (z + a)^{-(1/2)+\mathrm{i}\beta}(z - a)^{-(1/2)-\mathrm{i}\beta} \tag{87}$$

Since the magnetic induction and stresses are applied at infinity, the using of Eqs. (7), (8), (11), (17), (18), (28) and (59) for a cut free from surface tractions yields

$$P^+(t) = P^-(t) = 0, \quad A_{12}^+(t) = A_{12}^-(t) = 0 \tag{88}$$

Applying the following approximations

$$\frac{1}{X^+(t)} = t - 2\mathrm{i}\beta a + \mathrm{O}\left(\frac{1}{t}\right), \qquad \frac{1}{t-z} = \frac{1}{t} + \frac{z}{t^2} + \mathrm{O}\left(\frac{1}{t^3}\right) \quad \text{for } |t| \gg 1 \tag{89}$$

and using Eqs. (11), (37), (38), (66), (67) and (88), we have

$$\frac{X(z)}{2\pi\mathrm{i}} \int_L \frac{f(t)}{X^+(t)(t-z)} \mathrm{d}t = -\frac{X(z)}{2\pi\mathrm{i}} \frac{G_1 G_2}{G_2 + \kappa_2 G_1} \int_C \frac{B_{12}(t)}{X^+(t)(t-z)} \mathrm{d}t = (z - 2\mathrm{i}\beta a)U \tag{90}$$

$$\frac{1}{2\pi\mathrm{i}} \int_L \frac{g(t)}{t-z} \mathrm{d}t = -\frac{1}{2\pi\mathrm{i}} \frac{G_1 G_2}{G_2 + \kappa_2 G_1} \int_C \frac{B_{12}(t)}{(t-z)} \mathrm{d}t = U \tag{91}$$

where C is a close loop surrounding L and

$$U = -\frac{1}{2\mu_0} \frac{G_1 G_2}{G_2 + \kappa_2 G_1} \left\{ \left[2\chi_1 \left(\frac{1}{\lambda_1 + 2G_1} - \frac{1}{\lambda_2 + 2G_2} \right) + \frac{G_2 - G_1}{G_1 G_2} \right] B_{0x}^2 \right. \\ \left. + \left[2\chi_1 \left(\frac{1}{\mu_{r_1}^2} \frac{1}{\lambda_1 + 2G_1} - \frac{1}{\mu_{r_2}^2} \frac{1}{\lambda_2 + 2G_2} \right) + \frac{G_1 \mu_{r_1}^2 - G_2 \mu_{r_2}^2}{\mu_{r_1}^2 \mu_{r_2}^2 G_1 G_2} \right] B_{0y}^2 + \mathrm{i} \frac{2(G_2 \mu_{r_2} - G_1 \mu_{r_1})}{\mu_{r_1} \mu_{r_2}^2} B_{0x} B_{0y} \right\} \tag{92}$$

The coefficients s_1 and e_0 can be determined by substituting Eqs. (79), (80), (90) and (91) into Eqs. (77) and (78) and comparing with Eqs. (81) and (82). This gives

$$s_1 = \Gamma + \overline{\Gamma} + \overline{\Gamma'} + \frac{1}{2\mu_0} \left(1 - \frac{4\chi_2}{\mu_{r_2}^2} \right) (B_{0y}^2 - \mathrm{i} B_{0x} B_{0y}) - U \tag{93}$$

$$e_0 = \Gamma - \alpha(\overline{\Gamma} + \overline{\Gamma'}) + \frac{1}{8\mu_0} \left(1 - \frac{4\chi_2}{\mu_{r_2}^2} \right) [(1+\alpha) B_{0x}^2 + 4\mathrm{i}\alpha B_{0x} B_{0y} + (1 - 3\alpha) B_{0y}^2] - U \tag{94}$$

Eq. (86) for the requirement of single-valued displacement leads

$$\int_C \left[\kappa_1 (G_2 + \kappa_2 G_1) \Phi_2(\zeta) + (G_1 + \kappa_1 G_2) \Omega_2(\zeta) + \frac{\mu_0 G_1}{2} \overline{h_2'}(\zeta) \overline{h_2'}(\zeta) + \frac{\mu_0 G_2 \kappa_1}{2} \overline{h_1'}(\zeta) \overline{h_1'}(\zeta) \right. \\ \left. - \frac{\mu_0 \chi_2 G_1 G_2 \kappa_1}{\lambda_2 + 2G_2} h_2'(\zeta) \overline{h_2'}(\zeta) - \frac{\mu_0 \chi_1 G_1 G_2}{\lambda_1 + 2G_1} h_1'(\zeta) \overline{h_1'}(\zeta) \right] \mathrm{d}\zeta = 0 \tag{95}$$

Thus, the insertion of Eqs. (11), (37), (38), (77) and (78) into (95) yields

$$s_0 + U = -2\mathrm{i}\beta a (s_1 + U) \tag{96}$$

The complex functions $F_1(z)$ and $F_2(z)$ then can be rearranged as

$$F_1(z) = X(z) s_1^* (z - 2\mathrm{i}\beta a) \tag{97}$$

$$F_2(z) = e_0^* \tag{98}$$

where

$$s_1^* = \Gamma + \overline{\Gamma} + \overline{\Gamma'} + \frac{1}{2\mu_0}\left(1 - \frac{4\chi_2}{\mu_{r_2}^2}\right)(B_{0y}^2 - \mathrm{i}B_{0x}B_{0y}) \approx \Gamma + \overline{\Gamma} + \overline{\Gamma'} + \frac{1}{2\mu_0}(B_{0y}^2 - \mathrm{i}B_{0x}B_{0y}) \tag{99}$$

$$\begin{aligned} e_0^* &= \Gamma - \alpha(\overline{\Gamma} + \overline{\Gamma'}) + \frac{1}{8\mu_0}\left(1 - \frac{4\chi_2}{\mu_{r_2}^2}\right)[(1+\alpha)B_{0x}^2 + 4\mathrm{i}\alpha B_{0x}B_{0y} + (1-3\alpha)B_{0y}^2] \\ &\approx \Gamma - \alpha(\overline{\Gamma} + \overline{\Gamma'}) + \frac{1}{8\mu_0}[(1+\alpha)B_{0x}^2 + 4\mathrm{i}\alpha B_{0x}B_{0y} + (1-3\alpha)B_{0y}^2] \end{aligned} \tag{100}$$

It is noted that the property $\mu_{r_2} = \chi_2 + 1 \gg 1$ of soft ferromagnetic materials has been used in the final approximations of these equations. Having completed the solution of $F_1(z)$ and $F_2(z)$ in Eqs. (97) and (98), we can obtain the functions $\Phi_j(z)$ and $\Omega_j(z)$ $(j = 1, 2)$ in Eqs. (75)–(78), the magnetoelastic stresses and the Maxwell stress in Eq. (50) explicitly. In the special case for homogeneous material, i.e. $\alpha = 1$, the complex functions and stresses obtained here reduce to that given by Lin and Yeh (2002). For another special case of pure elastic problem by dropping all magnetic terms, the degenerated stresses are identical to that provided by Rice and Sih (1965). By the use of Eqs. (37), (38), (50)–(52), (75)–(78), (97) and (98), the magnetoelastic stresses on both side of the bonded surface are

$$\begin{aligned} (t_{yy} - \mathrm{i}t_{xy})_1 = {} & \frac{s_1^*(z - 2\mathrm{i}\beta a)}{\sqrt{z^2 - a^2}}\left(\frac{z+a}{z-a}\right)^{\mathrm{i}\beta} + \frac{1}{2\mu_0\mu_{r_2}^2}\left\{B_{0x}^2 + \left[4\chi_2 - (2\chi_1 + 1)\frac{\mu_{r_2}^2}{\mu_{r_1}^2}\right]\frac{B_{0y}^2 z^2}{z^2 - a^2} \right. \\ & \left. - \mathrm{i}2(\chi_2 - 1)\frac{B_{0x}B_{0y}z}{\sqrt{z^2 - a^2}}\right\} \quad \text{for } z \in L^* \end{aligned} \tag{101}$$

and

$$\begin{aligned} (t_{yy} - \mathrm{i}t_{xy})_2 = {} & \frac{s_1^*(z - 2\mathrm{i}\beta a)}{\sqrt{z^2 - a^2}}\left(\frac{z+a}{z-a}\right)^{\mathrm{i}\beta} + \frac{1}{2\mu_0\mu_{r_2}^2}\left\{B_{0x}^2 + (2\chi_2 - 1)\frac{B_{0y}^2 z^2}{z^2 - a^2} \right. \\ & \left. - \mathrm{i}2(\chi_2 - 1)\frac{B_{0x}B_{0y}z}{\sqrt{z^2 - a^2}}\right\} \quad \text{for } z \in L^* \end{aligned} \tag{102}$$

Thus, the total stresses on the bonded surface are found to be

$$\begin{aligned} (t_{yy} - \mathrm{i}t_{xy})_1^T &= (t_{yy} - \mathrm{i}t_{xy})_2^T \\ &= \frac{1}{\sqrt{z^2 - a^2}}\left[s_1^*(z - 2\mathrm{i}\beta a)\left(\frac{z+a}{z-a}\right)^{\mathrm{i}\beta} - \frac{2\mathrm{i}\chi_2 B_{0y}}{\mu_0\mu_{r_2}^2}\left(B_{0x}z + \mathrm{i}\frac{B_{0y}z^2}{\sqrt{z^2 - a^2}}\right)\right] \quad \text{for } z \in L^* \end{aligned} \tag{103}$$

which satisfy Eq. (46) then can be used to guarantee the exactness of the present solution. From Eqs. (101) and (102), we find that the tangential magnetoelastic stresses are continuous across the bonded surface but larger normal magnetoelastic stress t_{yy} appears in the surface of smaller magnetic permeability (susceptibility) material. The substitutions of $(t_{yy})_j^T$ obtained in Eqs. (103) and (50) into (48) give rise to

$$(t_{xx})_2^T = \eta(t_{xx})_1^T + \frac{(3+\eta)\alpha - (3\eta+1)}{1+\alpha} t_{yy}^T + CB_{0x}^2 + D\frac{B_{0y}^2 z^2}{z^2 - a^2} \tag{104}$$

where

$$\eta = \frac{G_2(\kappa_1+1)}{G_1(\kappa_2+1)} \tag{105}$$

and

$$C = \frac{1}{\mu_0\mu_{r_2}^2}\left[\frac{2\kappa_2^2+5\kappa_2+1}{(1+\kappa_2)^2}\chi_2 - \frac{G_2}{G_1}\frac{2\kappa_1^2+5\kappa_1+1}{(1+\kappa_1)(1+\kappa_2)}\chi_1 + \frac{1}{2(1+\kappa_2)}\left(1-\frac{G_2}{G_1}\right)\right] \tag{106}$$

$$D = \frac{1}{\mu_0\mu_{r_2}^2(1+\kappa_2)}\left[\frac{G_2(1+\kappa_2)(2\kappa_2-3) - G_1(\kappa_2-1)}{2G_1(1+\kappa_2)}\chi_2 + \frac{1}{2}\right] - \frac{G_2}{\mu_0\mu_{r_1}^2 G_1(1+\kappa_2)}\left(\frac{2\kappa_1^2+3\kappa_1-1}{1+\kappa_1}\chi_1 - \frac{1}{2}\right) \tag{107}$$

This equation indicates the jump of $(t_{xx})^T$ which is similar to that found by Rice and Sih (1965) for pure elastic loading.

The stress intensity factors at $z = a$ are defined as (Rice and Sih, 1965)

$$k_1 - \mathrm{i}k_2 = 2\sqrt{2}\mathrm{e}^{\pi\beta}\lim_{z\to a}(z-a)^{\frac{1}{2}+\mathrm{i}\beta}\Phi_1(z) \tag{108}$$

where

$$k_1 = \frac{(s_{1R}^* + 2\beta s_{1I}^*)\cos(\beta\ln 2a) + (2\beta s_{1R}^* - s_{1I}^*)\sin(\beta\ln 2a)}{\cosh\pi\beta}\sqrt{a} \tag{109}$$

$$k_2 = \frac{(2\beta s_{1R}^* - s_{1I}^*)\cos(\beta\ln 2a) - (s_{1R}^* + 2\beta s_{1I}^*)\sin(\beta\ln 2a)}{\cosh\pi\beta}\sqrt{a} \tag{110}$$

Here the notations s_{1R}^* and s_{1I}^* are the real and imaginary parts of s_1^*. The stress intensity factors that defined in Eq. (108) are introduced to measure the local energy intensification in the vicinity of crack tips. As referred to the stresses $(t_{yy})_j$ $(j = 1, 2)$ in Eqs. (101) and (102), the term $B_{0y}^2 z^2/(z^2 - a^2)$ of $1/r$ singularity is negligible as compared to those terms of $1/\sqrt{r}$ in the measurable range. Here r is the distance measured from crack tip as shown in Fig. 2. Therefore, the definition in Eq. (108) has presented the dominant singular behavior in the vicinity of crack tip. Such a result is similar to that remarked by Lin and Yeh (2002) for homogeneous medium and will be illustrated in the following paragraph. For special case of a homogeneous medium under remote uniform magnetic induction, the stress intensity factors can be obtained by taking $\alpha = 1$ in Eq. (68), $\Gamma = \Gamma' = 0$ in Eq. (83) and using Eqs. (73) and (99). It yields

$$(k_{\mathrm{I}} - \mathrm{i}k_{\mathrm{II}}) \approx \frac{\sqrt{a}}{2\mu_0}(B_{0y}^2 - \mathrm{i}B_{0x}B_{0y}) \tag{111}$$

which is consistent with that derived by Lin and Yeh (2002). For another special case of bounded dissimilar media under pure mechanical loading, the results presented here also can be reduced to that found by Rice and Sih (1965) by dropping all the terms related to magnetic fields. It is worthy to mention that several earlier authors, such as Shindo (1977) and Asanyan (1988), have considered the perturbed magnetic field induced by the coupling between the deformations and the magnetic fields in the undeformed state. The perturbed magnetic fields possess singularity of $1/\sqrt{r}$ but the magnetic fields in the original undeformed state were given to be uniform. Therefore, the perturbed fields are significant in the vicinity of crack tips despite of the assumption that they are much smaller than the original field. Even if the perturbed magnetic fields have the same order of singularity as those in Eqs. (39) and (40), they are much smaller than the magnetic fields obtained here then can be negligible in the present study.

The crack opening condition can be formulated as

$$\int_{-a}^{t} u_r'(t')\,\mathrm{d}t' \geqslant 0 \quad \text{for } t \in L \tag{112}$$

Here the kernel $u_r'(t)$ is the difference between the displacement derivatives of upper and lower surfaces. For the present problem, it is

$$u_r'(t) = \mathrm{Im}\{[u_x'(t) + \mathrm{i}u_y'(t)]^+ - [u_x'(t) + \mathrm{i}u_y'(t)]^-\} \tag{113}$$

where $[u_x'(t) + \mathrm{i}u_y'(t)]^+$ is defined in Eq. (84). Referring to the derivations of Eqs. (85) and (86) and making use of Eqs. (73), (75)–(78), (97) and (98), we find the crack opening condition in Eq. (112) is equivalent to

$$\mathrm{Im}\left[\frac{(G_1+\kappa_1 G_2)s_1^*}{2G_1G_2\alpha}(t'+a)^{\frac{1}{2}+\mathrm{i}\beta}(t'-a)^{\frac{1}{2}-\mathrm{i}\beta}\right]\Bigg|_{-a}^{t} = \frac{G_1+\kappa_1G_2}{2G_1G_2\sqrt{\alpha}}|s_1^*|\sqrt{a^2-t^2}\cos\left(\beta\log\frac{a+t}{a-t}+\theta\right) > 0 \quad \text{for } |t|\leqslant a \tag{114}$$

where $|s_1^*| = \sqrt{s_{1R}^{*2}+s_{1I}^{*2}}$ and $\theta = \tan^{-1}(s_{1I}^*/s_{1R}^*)$ denote the amplitude and argument of s_1^*. The crack opening condition in Eq. (114) leads

$$\frac{1}{\exp(\frac{\theta+\mathrm{sgn}(\beta)\pi/2}{\beta})+1} \leqslant \frac{\delta}{2a} \leqslant \frac{1}{\exp(\frac{\theta-\mathrm{sgn}(\beta)\pi/2}{\beta})+1} \tag{115}$$

for the range $0 \leqslant \delta/2a \leqslant 1$ of interface crack. Here the symbol $\mathrm{sgn}(\beta)$ which indicates the sign of β is defined as + and − for $\beta \geqslant$ and <0, respectively. For the case that only the pure mechanical loads σ_{yy}^∞ and σ_{xy}^∞ are applied at infinity, the angle θ becomes $-\tan^{-1}(\tau_{xy}^\infty/\sigma_{yy}^\infty)$ and Eq. (114) in the vicinity of right tip (i.e. $t \to a$) is identical to that given by Rice (1988). It is convenient to introduce the distance $\delta(= t + a)$ measured from the left end of crack tip.

5. Numerical illustration and discussion

Since the main concern of the present paper is focused on the effect of magnetic induction, the effects of applied stresses σ_{xx}^∞, σ_{yy}^∞ and τ_{xy}^∞ that have been well studied are dropped in all the illustrative figures. In the absence of mechanical loading, the angle θ becomes to $\gamma - \pi/2$ with the incident angle of magnetic induction $\gamma = \tan^{-1}(B_{0y}/B_{0x})$ as shown in Fig. 2. Thus, the variation of the range of $\delta/2a$ in Eq. (115) on the incident angle γ for various α are plotted in Fig. 3a and b. In these figures, the range of α should lying within the physically practical range $1/3 < \alpha < 3$ as derived by England (1965). For each value of α, the upper bound for available range of $\delta/2a$ is provided in Fig. 3a and the lower bound is given in Fig. 3b. That is, the crack will close for the area above the curves in Fig. 3a or below the curves in Fig. 3b for each α. It is found that the crack surface near the tips will come into contact except that both materials have the same elastic properties (i.e. $\alpha = 1$). Nevertheless, the crack closing will occur only on a very restrictive region near the tips. From the practical viewpoint, we can assign a physical detectable scale to check the crack opening condition. For example, the dash lines of $1 - \delta/2a$ and $\delta/2a$ equal to 0.0005 as remarked in Fig. 3a and b

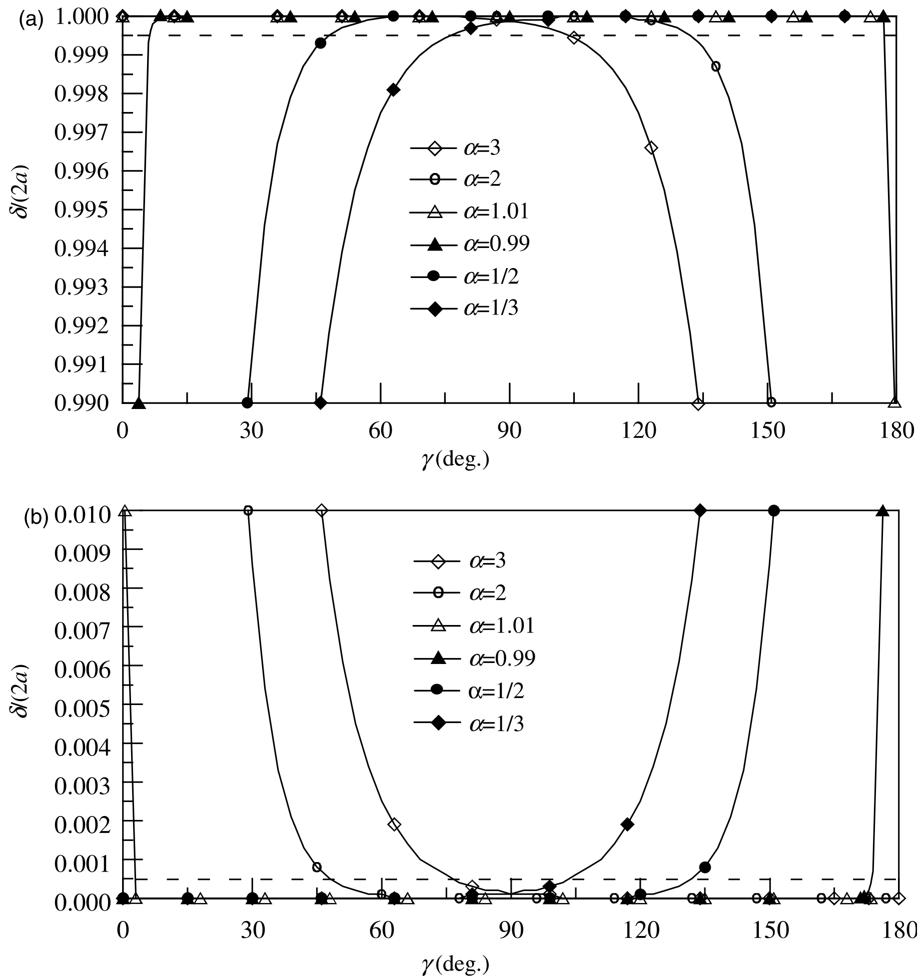

Fig. 3. The position of crack open condition under various incident angle γ and α: (a) upper bound and (b) lower bound.

can be adopted to examine the crack opening. The critical incident angle γ_{cr1} is obtained on the interaction of each curve with the dashed line. For each curve, the part beyond the dashed line in Fig. 3a and below the dashed in Fig. 3b are unavailable due to that the crack opening condition is violated.

The comparison of magnetoelastic stress t_{yy} and Maxwell stress t_{yy}^M is displayed in Fig. 4. It is noted that the magnitude of the magnetoelastic stress depends on α but the Maxwell stress is not. Furthermore, the former is much higher than the latter in the moderate range due to that the Maxwell stress has $1/r$ singularity while leaving the tip a distance r and decays more rapidly than the magnetoelastic stress. Since the term $B_{0y}^2 z^2/(z^2 - a^2)$ of t_{yy} in Eqs. (101) and (102) has the same order of magnitude as t_{yy}^M, it is guaranteed that the definition of stress intensity factor in Eq. (108) is adequate as mentioned above.

The variation of magnetoelastic stresses t_{yy}, t_{xy} on the distance r are plotted in Figs. 5 and 6. In these figures, r is measured away from the right tip (i.e. $z = a$) of crack along positive real axis and the value of α is taking to be 2 for illustration. It is also noted that the stresses shown here are in dimensionless form by dividing with $B_0^2/2\mu_0$. The typical magnetic induction $B_0 = 1$ T will induced magnetic stress $B_0^2/2\mu_0 = 58$ psi

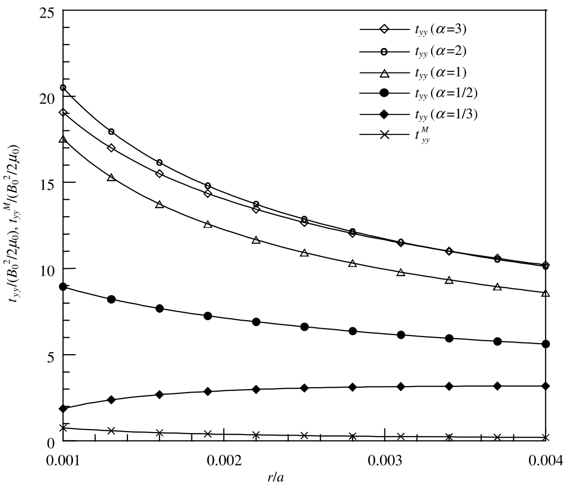

Fig. 4. The variation of non-dimensional magnetoelastic stress $t_{yy}/(B_0^2/2\mu_0)$ and Maxwell stress t_{yy}^M in bond with r.

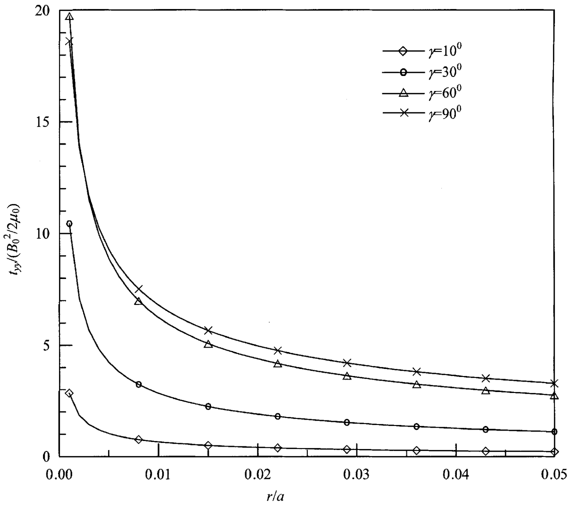

Fig. 5. The non-dimensional normal magnetoelastic stress $t_{yy}/(B_0^2/2\mu_0)$ in bond.

as remarked by Moon (1984). It is found that, when γ approaches to zero, the stresses decrease rapidly. Such a feature indicates that the component B_{0x} has no contribution on the singularity of magnetoelastic stress near the crack tip.

Figs. 7 and 8 display the variation of magnetoelastic stresses on the incident angle γ. Since negative t_{yy} may accompany the contact of the crack surfaces in the vicinity of tip, the parts of curves lying below $t_{yy} = 0$ are unavailable. Therefore, we can define the critical angle γ^*_{cr} as the intersection of the curves and the line of zero t_{yy}.

The values of γ_{cr} and γ^*_{cr} under various conditions are listed in Table 1. In which, the parameter δ' is defined as $2a - \delta$ to measure the distance from the right tip of crack as depicted in Fig. 2 and the values of

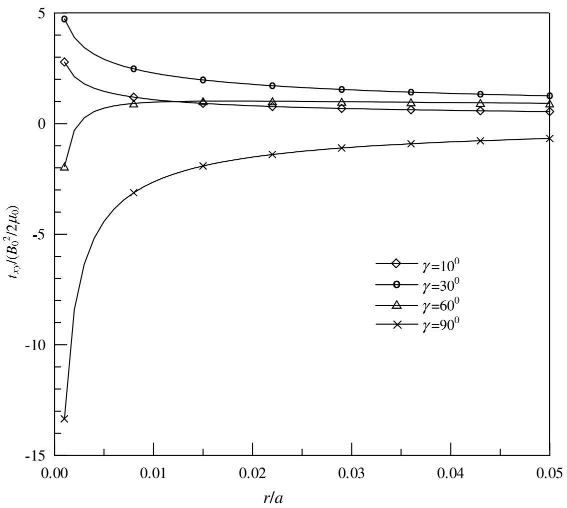

Fig. 6. The non-dimensional shear magnetoelastic stress $t_{xy}/(B_0^2/2\mu_0)$ in bond.

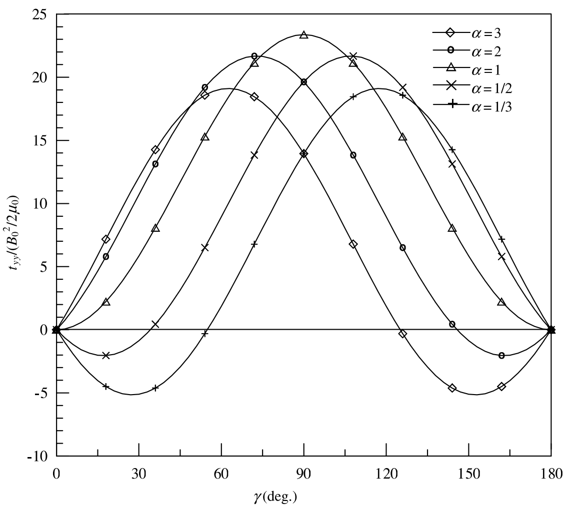

Fig. 7. The variation of non-dimensional magnetoelastic stress $t_{yy}/(B_0^2/2\mu_0)$ in bond with γ.

magnetic susceptibility χ_j ($j = 1, 2$) are taken to be 1000. This table provides a upper bound for the available range of γ for $\alpha > 1$ and lower bound for $\alpha < 1$. From this table, both the critical angles γ_{cr} and γ_{cr}^* monotonically increase with δ'/a or r/a, respectively. It is interpreted that the point of checking the crack opening or stress condition closer to the crack tip will lead more restricted available range.

Since the parameters δ' and r are the distance measured from crack tip toward the interior and exterior of crack, the critical angles γ_{cr} and γ_{cr}^* basing on different parameter cannot compare with each other. We find that γ_{cr}^* obtained from $r/a = 0.001$ is approximately equal to γ_{cr} that from $\delta'/a = 0.01$. It is remarked that γ_{cr} is used to confirm the crack opening then to provide the available range of angle γ for the present study. Nevertheless, an available scheme for incident angle of the magnetic induction is provided in Table 1.

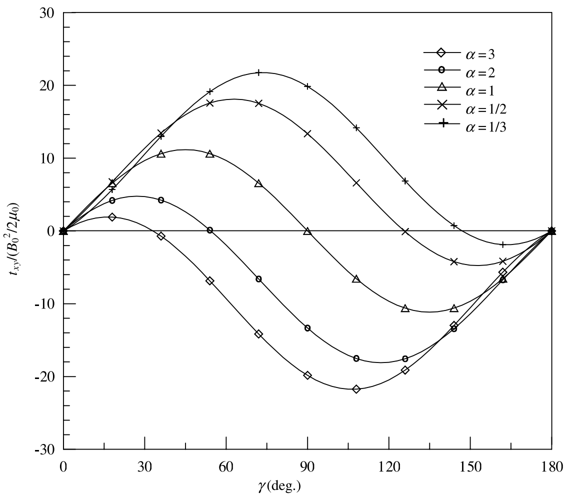

Fig. 8. The variation of non-dimensional magnetoelastic stress $t_{xy}/(B_0^2/2\mu_0)$ in bond with γ.

Table 1
The critical incident angle determined from displacement and stress conditions

Critical angle	γ_{cr} (°)			γ^*_{cr} (°)		
	$\delta'/a = 0.0001$	$\delta'/a = 0.001$	$\delta'/a = 0.01$	$r/a = 0.0001$	$r/a = 0.001$	$r/a = 0.01$
$\alpha = 3$	80.79	103.86	126.97	107.36	125.08	146.39
$\alpha = 2$	117.40	131.96	146.54	135.40	145.79	159.08
$\alpha = 1/2$	62.60	48.04	33.46	54.60	34.21	20.92
$\alpha = 1/3$	99.21	76.14	53.03	72.64	54.92	33.61

6. Conclusions

Basing upon the Hilbert problem formulation and the technique of analytic continuation, a general solution for the magnetoelastic problem of straight cracks in bonded dissimilar materials is obtained. In order to illustrate the application of the present study, detailed results are given for a single line crack case. By use of the solved magnetoelastic stress functions, the stress intensity factors near the crack tip and the crack opening condition are also provided. For different values of the ratio between the elastic properties of two materials, the magnetoelastic stress distribution has been displayed with figures. The trig-log feature of the magnetoelastic stresses is found just like that of pure elastic case. Comparison with the solution of the special case can guarantee the solution presented here is exact and general. The critical incident angle of magnetic induction should be obtained based on the requirement of crack opening assumption to define the available range of the present study. It is remarked that surface contact in a restricted range near the crack tip is unavoidable.

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